RADIAL AND NONRADIAL SOLUTIONS OF SOME ELLIPTIC SYSTEMS OF NONCOOPERATIVE TYPE

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Abstract. This paper is concerned with the following system of elliptic equations

$$\begin{cases} -\Delta u + u = F_u(|x|, u, v), \\ -\Delta v + v = -F_v(|x|, u, v), \\ \quad u, v \in H^1(\mathbb{R}^N). \end{cases}$$

It is shown that if F is odd in (u, v) and satisfy some growth conditions, then $(\mathcal{S})$ has infinitely many both radial and nonradial solutions. The proof relies on the Principle of Symmetric Criticality and a generalized Fountain Theorem for strongly indefinite functionals.

1. Introduction

We study the existence and multiplicity of solutions of the noncooperative elliptic system

$$(\mathcal{S}) \qquad \begin{cases} -\Delta u + u = F_u(x, u, v), \\ -\Delta v + v = -F_v(x, u, v), \\ \quad u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, and F_w designates the partial derivative of F with respect to w.

The natural energy associated to this problem is defined on the Hilbert space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ by

$$J(u, v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) - \int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) - \int_{\mathbb{R}^N} F(x, u, v). \tag{1}$$

This functional is strongly indefinite in the sense that its quadratic part has positive and negative eigenspaces both infinite-dimensional. Therefore some usual critical point theorems cannot be used. This is the first difficulty to overcome when investigating the existence of solutions of $(\mathcal{S})$. The second difficulty is the lack of the compactness of the Sobolev embeddings, since we consider the whole space. Fortunately when the problem has some symmetry properties, for example when it is invariant by a group of orthogonal transformations, it suffices to consider invariant functions to recover compactness.

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There are various methods in literature dealing with symmetric strongly indefinite functionals. Among others we cite Rabinowitz [12], Benci and Rabinowitz [5], Benci [4], Li [9], Bartsch and Clapp [1] and Batkam and Colin [3].

In this paper we assume that $(\mathcal{S})$ is invariant under the actions of the groups $\mathbb{Z}_2$ and $\mathcal{O}(N)$, and we consider the existence of infinitely many radial and nonradial solutions. Inspired by Bartsch and willem [2], our approach is based on the Principle of Symmetric Criticality of Palais [11] and a generalized Fountain Theorem for strongly indefinite even functionals in [3].

Before we state the main results, we introduce the following conditions:

(F_1) $F \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x,0,0) = 0$ $\forall x \in \mathbb{R}^N$.

(F_2) $|F_u(x,u,v)| + |F_v(x,u,v)| \leqslant c\big(|u| + |v| + |u|^{p-1} + |v|^{p-1}\big)$, with $2 < p < 2^*$.

(F_3) $\exists \gamma > 2$ such that $0 < \gamma F(x,u,v) \leqslant uF_u(x,u,v) + vF_v(x,u,v)$, $\forall (u,v) \neq (0,0)$.

(F_4) $\inf\big\{F(x,u,v) \,\big|\, |(u,v)| \geqslant 1,\, x \in \mathbb{R}^N\big\} > 0$.

(F_5) $|F_u(x,u,v)| + |F_v(x,u,v)| = \circ(|(u,v)|)$, $|(u,v)| \to \infty$ uniformly on $\mathbb{R}^{\mathbb{N}}$.

(F_6) $vF(x,u,v) \geqslant 0$ $\forall x \in \mathbb{R}^N$, $\forall (u,v) \in \mathbb{R}^2$.

(F_7) $F(x,u,v) = F(|x|,u,v)$, $\forall x \in \mathbb{R}^N$, $\forall (u,v) \in \mathbb{R}^2$.

(F_8) $F(x,-u,-v) = F(x,u,v)$.

The main results are the following:

Theorem 1. *If F satisfies the assumptions $(F_1) - (F_8)$, then $(\mathcal{S})$ has a sequence of radial solutions (u_k, v_k) such that*

$$\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u_k|^2 + |u_k|^2 - |\nabla v_k|^2 - |v_k|^2\big) - \int_{\mathbb{R}^N} F(x,u_k,v_k) \to +\infty, \quad k \to \infty.$$

Theorem 2. *Let $N = 4$ or $N \geqslant 6$. If F satisfied $(F_1) - (F_8)$, then $(\mathcal{S})$ has a sequence (y_k, z_k) of nonradial solutions such that*

$$\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla y_k|^2 + |z_k|^2 - |\nabla y_k|^2 - |z_k|^2\big) - \int_{\mathbb{R}^N} F(x,y_k,z_k) \to +\infty, \quad k \to \infty.$$

These theorems was first proved by Huang and Li [7] by using the Principle of Symmetric Criticality and the Limit Index Theory due to Li [9]. In this paper we propose a quite different approach which is much simpler.

Our main results are proved in Section 3, while the require abstract materials for the proofs are given is section 2.

2. Preliminaries

In this section we recall the abstract results we will use in the proofs of the main theorems.

2.1. Principle of symmetric criticality.

Definition 3. The action of a topological group G on a normed vector space X is a continuous map

$$G \times X \to X, \qquad (g,u) \mapsto g \cdot u$$

such that

$$1 \cdot u = u,\ (gh) \cdot u = g(h \cdot u) \text{ and the map } u \mapsto g \cdot u \text{ is linear, } \forall g, h \in G.$$

A subset A of X is invariant if $g \cdot A = A$, for every $g \in G$. A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi(g \cdot u) = \varphi(u)$, for every $u \in X$, $g \in G$.

We denote

$$Fix(G) := \big\{u \in X\ ;\ g \cdot u = u\ \forall g \in G\big\}$$

the set of invariant points.
The following result is due to Palais [11] (see also [13], Theorem 1.28):

Theorem 4 (Principle of symmetric criticality)**.** *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ is invariant and if u is a critical point of φ restricted to $Fix(G)$, then u is a critical point of φ.*

2.2. Generalized fountain theorem.

Now we assume that X is a separable Hilbert space. Let Y be a closed subspace of X. On $X = Y \oplus Y^\perp$ we consider the $\tau-$topology introduced by Kryszewski and Szulkin [8]; that is the topology associated to the norm

$$|||u||| := \max\Big(\sum_{j=0}^{\infty} |(Pu, a_j), \|Qu\||\Big), \quad u \in X,$$

where $(a_j)_{j\geqslant 0}$ is an orthonormal basis of Y, $P : X \to Y$ and $Q : X \to Z := Y^\perp$ are the orthogonal projections.
Consider an orthonormal basis $(e_j)_{j\geqslant 0}$ of Z and define

$$Y_k := Y \oplus (\oplus_{j=0}^{k} \mathbb{R} f_j), \qquad Z_k := \overline{\oplus_{j=k}^{\infty} \mathbb{R} f_j},$$

$$B_k := \big\{u \in Y_k \,\big|\, \|u\| \leqslant \rho_k\big\}, \quad N_k := \big\{u \in Z_k \,\big|\, \|u\| = r_k\big\} \text{ where } \ 0 < r_k < \rho_k,\ \ k \geqslant 1.$$

Definition 5. Let $\varphi \in \mathcal{C}^1(X,\mathbb{R})$

1. φ is said to satisfy the $(PS)_c$ condition $\big($or the Palais-Smale condition at level $c\big)$ if any sequence $(u_n) \subset X$ such that
$$\varphi(u_n) \to c \quad \text{ and } \quad \varphi'(u_n) \to 0$$
has a convergent subsequence.
2. We say that φ is $\tau-$upper semicontinuous if for every $C \in \mathbb{R}$ the set $\big\{u \in X\ ;\ J(u) \geqslant C\big\}$ is $\tau-$closed.
3. We say that $\nabla\varphi$ is weakly sequentially continuous if the sequence $(\nabla\varphi(u_n))$ converges to $\nabla\varphi(u)$ whenever (u_n) converges to u in X.

The following result is due to Batkam and Colin [3]:

Theorem 6 (Generalized fountain theorem)**.** *Let $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ be an even functional which is τ-upper semicontinuous and such that $\nabla\varphi$ is weakly sequentially continuous. If, for every $k \geqslant k_0$, there exist $\rho_k > r_k > 0$ such that:*

(A_1) $a_k := \sup_{\substack{u\in Y_k\\ \|u\|=\rho_k}} \varphi(u) \leqslant 0$ and $\sup_{\substack{u\in Y_k\\ \|u\|\leqslant\rho_k}} \varphi(u) < \infty$.

(A_2) $b_k := \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} \varphi(u) \to \infty$, $k \to \infty$.

(A_3) *φ satisfies the $(PS)_c$ condition, $\forall c > 0$.*

Then φ has an unbounded sequence of critical values.

3. Proof of the main results

Throughout this section $|\cdot|_p$ stands for the usual L^p norm, and $\|\cdot\|$ stands for the usual H^1 norm. We denote $\to$ (resp. $\rightharpoonup$) the strong convergence (resp. the weak convergence).

Definition 7. *Let $1 \leqslant p, q < \infty$. On the space $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ we define the norm*

$$|u|_{p\wedge q} = |u|_p + |v|_q.$$

On the space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ we define the norm

$$|u|_{p\vee q} = \inf\big\{|v|_p + |w|_q;\ \ v \in L^p(\mathbb{R}^N),\, w \in L^q(\mathbb{R}^N),\ \ u = v + w\big\}.$$

We refer to [7] for the proof of the following lemma:

Lemma 8. *Assume that $1 \leqslant p, q, r, s < \infty$, $H \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^2)$ and*

$$|H(x, u, v)| \leqslant C\big(|(u, v)|^{\frac{p}{r}} + |(u, v)|^{\frac{q}{s}}\big).$$

Then, for every $u, v \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, $H(\cdot, u, v) \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ and the operator

$$T : \big(L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)\big)\times\big(L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)\big) \to L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N) :\ (u, v) \mapsto H(x, u, v)$$

is continuous.

Lemma 9. *$J \in \mathcal{C}^1(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \mathbb{R})$ with*

$$\big\langle J'(u, v), (\phi, \varphi)\big\rangle = (u, \phi)_1 - (v, \varphi)_1 - \int_{\mathbb{R}^N} \big(\phi F_u(x, u, v) + \varphi F_v(x, u, v)\big), \tag{2}$$

where $(\cdot, \cdot)_1$ denotes the usual inner product of $H^1(\mathbb{R}^N)$.

Proof. **Existence of the Gateaux derivative.** Let $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. For every $\phi, \varphi \in H^1(\mathbb{R}^N)$ and for $0 < |t| < 1$ we have:

$$\begin{aligned}\frac{1}{t}\big[J(u+t\phi, v+t\varphi)-J(u, v)\big] = &\int_{\mathbb{R}^N} \Big(\nabla u\nabla\phi+u\phi-\nabla v\nabla\varphi-v\varphi+\frac{t}{2}(|\nabla\phi|^2+|\phi|^2-|\nabla\varphi|^2-|\varphi|^2)\Big)\\ &- \int_{\mathbb{R}^N} \frac{1}{t}\big(F(x, u + t\phi, v + t\varphi) - F(x, u, v)\big).\end{aligned}$$

It follows from the mean value theorem that there exists $\lambda \in (0,1)$ such that, given $x \in \mathbb{R}^N$

$$\begin{aligned}
\frac{1}{|t|}|F(x,u+t\phi,v+t\varphi)-F(x,u,v)| &\leqslant |F_u(x,u+t\lambda\phi,v+t\lambda\varphi)||\phi| \\
&\quad + |F_v(x,u+t\lambda\phi,v+t\lambda\varphi)||\varphi| \\
&\leqslant (|F_u(x,u+t\lambda\phi,v+t\lambda\varphi)|+|F_v(x,u+t\lambda\phi,v+t\lambda\varphi)|)(|\phi|+|\varphi|) \\
&\leqslant c(|\lambda\phi|+|v+t\lambda\varphi|+|\lambda\phi|^{p-1}+|v+t\lambda\varphi|^{p-1})(|\phi|+|\varphi|) \quad \text{(in view of } (F_2)) \\
&\leqslant c\big(|u|+|\phi|+|v|+|\varphi|+2^{p-1}(|u|^{p-1}+|\phi|^{p-1})+2^{p-1}(|v|^{p-1}+|\varphi|^{p-1})\big)(|\phi|+|\varphi|).
\end{aligned}$$

The Hölder inequality implies that

$$c\big(|u|+|\phi|+|v|+|\varphi|+2^{p-1}(|u|^{p-1}+|\phi|^{p-1})+2^{p-1}(|v|^{p-1}+|\varphi|^{p-1})\big)(|\phi|+|\varphi|) \in L^1(\mathbb{R}^N).$$

It then follows from the dominated convergence theorem that

$$\big\langle J'(u,v),(\phi,\varphi)\big\rangle = (u,\phi)_1 - (v,\varphi)_1 - \int_{\mathbb{R}^N} \big(\phi F_u(x,u,v)+\varphi F_v(x,u,v)\big).$$

Continuity of the derivative. Let $(u_n,v_n) \subset H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ such that $(u_n,v_n) \to (u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. By the Sobolev embedding theorem $(u_n,v_n) \to (u,v)$ in $\big(L^2(\mathbb{R}^N \cap L^p(\mathbb{R}^N)\big) \times \big(L^2(\mathbb{R}^N \cap L^p(\mathbb{R}^N)\big)$. By Lemma 8 $F_u(x,u_n,v_n) \to F_u(x,u,v)$ and $F_v(x,u_n,v_n) \to F_v(x,u,v)$ in $L^2(\mathbb{R}^N)+L^{p'}(\mathbb{R}^N)$. The Hölder inequality implies that

$$\begin{aligned}
|\big\langle J'(u_n,v_n)-J'(u,v)\big\rangle| &\leqslant \|u_n-u\|\|\phi\| + \|v_n-v\|\|\varphi\| \\
&\quad + |F_u(x,u_n,v_n)-F_u(x,u,v)|_{2\vee p'}|\phi|_{2\wedge p} \\
&\quad + |F_v(x,u_n,v_n)-F_v(x,u,v)|_{2\vee p'}|\varphi|_{2\wedge p}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|J'(u_n,v_n)-J'(u,v)\| &\leqslant \|u_n-u\|\| + \|v_n-v\| \\
&\quad + C\big[|F_u(x,u_n,v_n)-F_u(x,u,v)|_{2\vee p'} + |F_v(x,u_n,v_n)-F_v(x,u,v)|_{2\vee p'}\big].
\end{aligned}$$

We then deduce that $J'(u_n,v_n)-J'(u,v) \to 0$ as $n \to \infty$. □

3.1. **Existence of radial solutions.** We recall that the action of the group $\mathcal{O}(N)$ on $H^1(\mathbb{R}^N)$ is defined by

$$(g\cdot u)(x) = u(g^{-1}x).$$

Let $H^1_{\mathcal{O}(N)}(\mathbb{R}^N) := \big\{u \in H^1(\mathbb{R}^N)\,\big|\, g\cdot u = u,\ \forall g \in \mathcal{O}(N)\big\}$ the set of radial functions. The following result is due to Strauss:

Lemma 10 (Strauss, 1977)**.** *Let $N \geqslant 2$. For $2 < q < 2^\star$, the embedding $H^1_{\mathcal{O}(N)}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact.*

We define

$$X := H^1_{\mathcal{O}(N)}(\mathbb{R}^N)\times H^1_{\mathcal{O}(N)}(\mathbb{R}^N) \quad \text{and} \quad \Phi := J|_X.$$

By (F_7) J is invariant under the action of $\mathcal{O}(N)$. It then follows from the Principle of Symmetric Criticality (Lemma 4) that the critical points of Φ are weak solutions of $(\mathcal{S})$.

Lemma 11. *Φ satisfies the Palais-Smale condition on X. That is, every sequence $(u_n,v_n)\subset X$ such that $(\Phi(u_n,v_n))$ is bounded and $\Phi'(u_n,v_n)\to 0$, has a convergent subsequence.*

Proof. Let $(u_n,v_n)\subset X$ such that

$$d:=\sup_n|\Phi(u_n)|<\infty \quad\text{and}\quad \Phi'(u_n,v_n)\to 0 \quad\text{as}\quad n\to\infty.$$

We want to show that (u_n,v_n) has a convergent susbsequence.
By (2) and (F_6) we have

$$\big\langle -\Phi'(u_n,v_n),(0,v_n)\big\rangle=\|v_n\|^2+\int_{\mathbb{R}^N}v_nF_v(x,u_n,v_n)\geqslant\|v_n\|^2.$$

Hence for n big enough we have $\|v_n\|^2\leqslant\|v_n\|$. This shows that (v_n) is bounded.
On the other hand we deduce from (1) and (2) that

$$\begin{aligned}\Phi(u_n,v_n)-\frac{1}{\gamma}\big\langle\Phi'(u_n,v_n),(u_n,v_n)\big\rangle&=\big(\frac12-\frac1\gamma\big)\|u_n\|^2-\big(\frac12-\frac1\gamma\big)\|v_n\|^2\\&\quad+\int_{\mathbb{R}^N}\big[\gamma(u_nF_u(x,u_n,v_n)+v_nF_v(x,u_n,v_n))-F(x,u_n,v_n)\big]\\&\geqslant\big(\frac12-\frac1\gamma\big)\|u_n\|^2-\big(\frac12-\frac1\gamma\big)\|v_n\|^2\quad\big(\text{in view of }(F_3)\big).\end{aligned}$$

Hence for n big enough we have

$$\big(\frac12-\frac1\gamma\big)\|u_n\|^2-\big(\frac12-\frac1\gamma\big)\|v_n\|^2\leqslant\|(u_n,v_n)\|+d.$$

Since (v_n) is bounded, we deduce that (u_n) is also bounded.
Now up to a subsequence we have $(u_n,v_n)\rightharpoonup(u,v)$ in X. By Lemma 10, $u_n\to u$ and $v_n\to v$ in $L^P(\mathbb{R}^N)$.
We easily deduce from (1) and (2) that

$$\|u_n-u\|^2=\big\langle\Phi'(u_n,v_n)-\Phi'(u,v),(u_n-u,0)\big\rangle+\int_{\mathbb{R}^N}\big(F_u(x,u_n,v_n)-F_u(x,u,v)\big)(u_n-u)$$

$$\|v_n-v\|^2=-\big\langle\Phi'(u_n,v_n)-\Phi'(u,v),(0,v_n-v)\big\rangle-\int_{\mathbb{R}^N}\big(F_v(x,u_n,v_n)-F_v(x,u,v)\big)(v_n-v).$$

Clearly $\big\langle\Phi'(u_n,v_n)-\Phi'(u,v),(u_n-u,0)\big\rangle\to 0$ as $n\to\infty$.
(F_2) and (F_5) imply that for all $\varepsilon>0$ there exists $c_\varepsilon>0$ such that

$$|F_u(x,u,v)|+|F_v(x,u,v)|\leqslant\varepsilon(|u|+|v|)+c_\varepsilon(|u|^{p-1}+|v|^{p-1}).$$

This implies that

$$\begin{aligned}|\big(F_u(x,u_n,v_n)-F_u(x,u,v)\big)(u_n-u)|&\leqslant\varepsilon\big(|u_n|+|v_n|+|u|+|v|\big)(|u_n|+|u|)\\&\quad+c_\varepsilon\big(|u_n|^{p-1}+|u|^{p-1}+|v_n|^{p-1}+|v|^{p-1}\big)|u_n-u|.\end{aligned}$$

Since the sequence (u_n,v_n) is bounded in X, we obtain by using the Hölder inequality

$$\int_{\mathbb{R}^N}|\big(F_u(x,u_n,v_n)-F_u(x,u,v)\big)(u_n-u)|\leqslant C\big(\varepsilon+c_\varepsilon|u_n-u|_p\big),$$

where C is constant independent of ε and n. It is then easy to see that $u_n \to u$ as $n \to \infty$.
By the same way we show that $v_n \to v$ as $n \to \infty$. $\square$

Lemma 12. $\nabla\Phi$ *is weakly sequentially continuous.*

Proof. Let $(u_n, v_n) \subset X$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in X. (2) and the Hölder inequality imply, for any $\phi, \varphi \in X$

$$\begin{aligned}\big|\langle J'(u_n,v_n) - J'(u,v), (\phi,\varphi)\rangle\big| &\leqslant \Big|\int_{\mathbb{R}^N} \big(\nabla(u_n-u)\nabla\phi + (u_n-u)\phi\big)\Big| \\ &+ \Big|\int_{\mathbb{R}^N} \big(\nabla(v_n-v)\nabla\varphi + (v_n-v)\varphi\big)\Big| + \big|F_u(x,u_n,v_n) - F_u(x,u,v)\big|_{2\vee p'}|\phi|_{2\wedge p} \\ &+ \big|F_v(x,u_n,v_n) - F_v(x,u,v)\big|_{2\vee p'}|\varphi|_{2\wedge p}.\end{aligned}$$

It is clear that

$$\int_{\mathbb{R}^N} \big(\nabla(u_n-u)\nabla\phi + (u_n-u)\phi\big) \to 0 \text{ and } \int_{\mathbb{R}^N} \big(\nabla(v_n-v)\nabla\varphi + (v_n-u)\varphi\big) \to 0.$$

By Lemma 10, $u_n \to u$ and $v_n \to v$ in $L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. We then deduce from Lemma 8 that $F_u(x,u_n,v_n) - F_u(x,u,v) \to 0$ and $F_v(x,u_n,v_n) - F_v(x,u,v) \to 0$ in $L^2(\mathbb{R}^N) + L^{p'}(\mathbb{R}^N)$.
It then follows that $J'(u_n,v_n) \rightharpoonup J'(u,v)$. $\square$

We define

$$Y := \{0\} \times H^1_{\mathcal{O}(N)}(\mathbb{R}^N), \qquad Z := H^1_{\mathcal{O}(N)}(\mathbb{R}^N) \times \{0\},$$

and we consider the $\tau-$topology on $X = Y \oplus Z$.

Lemma 13. Φ *is $\tau-$upper semicontinuous.*

Proof. Let $(u_n,v_n) \subset X$ and $C \in \mathbb{R}$ such that $(u_n,v_n) \xrightarrow{\tau} (u,v)$ in X and $J(u_n,v_n) \geqslant C$. By the definition of τ, $u_n \to u$ in $H^1_{\mathcal{O}(N)}(\mathbb{R}^N)$.

$$J(u_n,v_n) \geqslant C \Longleftrightarrow \frac{1}{2}\|v_n\|^2 + C \leqslant \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n,v_n).$$

Since $F \geqslant 0$ we deduce that

$$\frac{1}{2}\|v_n\|^2 + C \leqslant \frac{1}{2}\|u_n\|^2.$$

Since (u_n) is bounded, we easily deduce that (v_n) is also bounded.
Now we may suppose, up to a subsequence that

$$\begin{aligned}(u_n,v_n) &\rightharpoonup (u,v) \text{ in } X, \\ u_n(x) &\to u(x), \quad v_n(x) \to v(x) \text{ } a.e. \text{ in } \mathbb{R}^N, \\ F(x,u_n(x),v_n(x)) &\to F(x,u(x),v(x)) \text{ } a.e. \text{ in } \mathbb{R}^N.\end{aligned}$$

It follows from Fatou's lemma and the weak lower semicontinuity of the norm $\|\cdot\|$ that $-J(u,v) \leqslant -C$. $\square$

***Proof of Theorem** 1.* Let $(e_j)_{j\geqslant 0}$ be an orthonormal basis of $H^1_{\mathcal{O}(N)}(\mathbb{R}^N)$. Set for $k \geqslant 2$,

$$Y_k = Y \oplus \Big(\bigoplus_{j=0}^{k} \mathbb{R}e_j \times \{0\}\Big) \text{ and } Z_k = \overline{\bigoplus_{j=k}^{\infty} \mathbb{R}e_j} \times \{0\}.$$

Let $(u,v)\in Y_k$. By (F_3) and (F_4), for all $\delta>0$ there exists $C_1=C_1(\delta)>0$ such that $F(x,u,v)\geqslant C_1|(u,v)|^\gamma-\delta|(u,v)|^2$. This implies that

$$\Phi(u,v)\leqslant \frac{1}{2}\|u\|^2-\frac{1}{2}\|v\|^2+\delta\big(|u|^2+|v|^2\big)-C_1\big(|u|^\gamma+|v|^\gamma\big)$$
$$\leqslant \frac{1}{2}\|u\|^2-\frac{1}{2}\|v\|^2+\delta\big(|u|^2+|v|^2\big)-C_1|u|^\gamma.$$

Since all norms are equivalent on $\bigoplus_{j=0}^{k}\mathbb{R}e_j$, we obtain

$$\Phi(u,v)\leqslant\big(\frac{1}{2}+\delta\big)\|u\|^2-\big(\delta-\frac{1}{2}\big)\|v\|^2-C_2C_1\|u\|^\gamma,$$

where $C_2>0$ is a constant. By choosing $\delta<\frac{1}{4}$, we obtain

$$\Phi(u,v)\leqslant\frac{3}{4}\|u\|^2-\frac{1}{4}\|v\|^2-C\|u\|^\gamma.$$

Hence $\Phi(u,v)\to-\infty$ as $\|(u,v)\|\to+\infty$, and assumption (A_1) of Theorem 6 is then satisfied for ρ_k sufficiently large.

Now let $(u,0)\in Z_k$. Let $\varepsilon>0$, by (F_2) and F_3 there exists C_ε such that $F(x,u,0)\leqslant\varepsilon|u|^2+C_\varepsilon|u|^p$, which implies

$$\Phi(u,0)\geqslant\frac{1}{2}\|u\|^2-\varepsilon|u|^2-C_\varepsilon|u|^p\geqslant\big(\frac{1}{2}-\varepsilon\big)\|u\|^2-C_\varepsilon|u|^p.$$

By choosing $\varepsilon<\frac{1}{4}$ we obtain

$$\Phi(u,0)\geqslant\frac{1}{4}\|u\|^2-C|u|^p\geqslant\frac{1}{4}\|u\|^2-C\beta_k^p\|u\|^p,$$

where

$$\beta_k:=\sup_{\substack{w\in\overline{\bigoplus_{j=k}^{\infty}\mathbb{R}e^j}\\ \|w\|=1}}|w|_p.$$

Let

$$r_k:=\big(2pC\beta_k\big)^{\frac{1}{2-p}}.$$

Then for any $(u,0)\in Z_k$ such that $\|u\|=r_k$ we obtain

$$\Phi(u,0)\geqslant\big(\frac{1}{4}-\frac{1}{2p}\big)r_k^2\to\infty,\quad\text{as}\quad k\to\infty,$$

since $\beta_k\to0$ as $k\to\infty$ $\big($ see [13] Lemma 3.8$\big)$. The assumption (A_2) of Theorem 6 is satisfied.

By (F_8) Φ is even, and by Lemma 11 the assumption (A_3) of Theorem 6 is satisfied. We then conclude, in view of Lemmas 9, 12 and 13 by applying Theorem 6. □

3.2. **Existence of nonradial solutions.** Let $N=4$ or $N\geqslant6$. let $2\leqslant m\leqslant N/2$ an integer different from $(N-1)/2$. We recall that the action of the group $G:=\mathcal{O}(m)\times\mathcal{O}(m)\times\mathcal{O}(N-2m)$ on $H^1(\mathbb{R}^N)$ is defined by $(g\cdot u)(x)=u(g^{-1}x)$. Let

$$H^1_G(\mathbb{R}^N):=\big\{u\in H^1(\mathbb{R}^N)\,;\,g\cdot u=u\ \forall g\in G\big\}.$$

Lemma 14 (P. L. Lions, [10])**.** *For $2<q<2^*$, the following embedding $H^1_G(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$ is compact.*

Denote ι the involution defined on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{N-2m}$ by

$$\iota(x_1, x_2, x_3) := (x_2, x_1, x_3)$$

The action of $H := \{id_{\mathbb{R}^N}, \iota\}$ on $H^1_G(\mathbb{R}^N)$ is defined by

$$(h \cdot u)(x) = \begin{cases} u(x) & \text{if } h = id_{\mathbb{R}^N}, \\ -u(h^{-1}x) & \text{if } h = \iota. \end{cases}$$

By this construction, 0 is the only radial function in

$$H^1_{G,H}(\mathbb{R}^N) := \big\{u \in H^1_G(\mathbb{R}^N)\,;\, h \cdot u = u,\, \forall h \in H\big\}.$$

***Proof of Theorem** 2.* We define

$$X := H^1_{G,H}(\mathbb{R}^N) \times H^1_{G,H}(\mathbb{R}^N) \quad \text{and } \Psi := J|_X.$$

By (F_7) and the Principe of Symmetric Criticality (Theorem 4), the critical points of Ψ are also critical points of J.
Consider the $\tau-$topology on $X = Y \oplus Z$, where

$$Y := \{0\} \times H^1_{G,H}(\mathbb{R}^N) \text{ and } Z := H^1_{G,H}(\mathbb{R}^N) \times \{0\}.$$

The rest of the proof follows the lines of the proof of Theorem 1 above. □

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